AUTOMORPHISM GROUPS OF RIGHT-ANGLED BUILDINGS: SIMPLICITY AND LOCAL SPLITTINGS

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Abstract. We show that the group of type-preserving automorphisms of any irreducible semi-regular thick right-angled building is abstractly simple. When the building is locally finite, this gives a large family of compactly generated (abstractly) simple locally compact groups. Specializing to appropriate cases, we obtain examples of such simple groups that are locally indecomposable, but have locally normal subgroups decomposing non-trivially as direct products.

"Everywhere there was evidence of a collective obsession with the comforting logic of right angles."

(R. Larsen, The selected works of T.S. Spivet, 2009)

1. Introduction

Let (W, I) be a **right-angled** Coxeter system, i.e. a Coxeter system such that $m_{i,j} = 2$ or $m_{i,j} = \infty$ for all $i \neq j$. We assume that the generating set I is finite.

Haglund–Paulin have shown that for any tuple of (not necessarily finite) cardinalities $(q_i)_{i \in I}$, there exists a right-angled building of type (W, I) with **prescribed thicknesses** $(q_i)_{i \in I}$, in the sense that for each $i \in I$, all i-panels have thickness of the same cardinality q_i. We refer to [Dav98, Th. 5.1] for a group-theoretic construction of that building. Moreover, such a building is unique up to isomorphism (see Proposition 1.2 in [HP03]). A right-angled building satisfying that condition on the panels is called **semi-regular** (this terminology is motivated by the case of trees). It is **thick** if $q_i > 2$ for all $i \in I$.

The following shows that the automorphism groups of these buildings provide a large family of simple groups.

Theorem 1.1. *Let X be a thick semi-regular building of right-angled type (W, I). Assume that (W, I) is irreducible non-spherical.*

Then the group $\mathrm{Aut}(X)^+$ of type-preserving automorphisms of X is abstractly simple, and acts strongly transitively on X.

Recall that **strong transitivity** means transitivity on pairs (c, A) consisting of a chamber c and an apartment A containing c (we implicitly refer to the complete apartment system). Haglund and Paulin [HP03, Prop. 1.2] have shown that $\mathrm{Aut}(X)^+$ is chamber-transitive; in fact, the main tools in the proof of Theorem 1.1 rely on their work in an essential way.

If W is infinite dihedral, then a building X of type (W, I) with prescribed thicknesses $(q_i)_{i \in I}$ is nothing but a semi-regular tree. In that case the simplicity of the type-preserving automorphism group $G = \mathrm{Aut}(X)^+$ is due to Tits [Tit70]. If (W, I) is a **right-angled Fuchsian group** (i.e. if $I = \{1, \ldots, r\}$ and $m_{ij} = 2$ if and only

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if $|i-j|=1$ or $r-1$), then a building X of type (W,I) is a **Bourdon building**, and the simplicity statement is due to Halgund–Paulin [HP98].

After this work was completed, K. Tent informed me that she had obtained independently a proof of *bounded* simplicity in the case right-angled buildings whose panels are of countable thickness; this stronger simplicity statement means that there is a uniform constant N such that the group can be written as a product of N copies of each of its non-trivial conjugacy class. In case of trees, bounded simplicity was proved without any restriction on the thickness by J. Gismatullin [Gis09]. Another related simplicity theorem was also obtained by N. Lazarovich [Laz]; it applies to a large family of groups acting on locally finite, finite-dimensional CAT(0) cube complexes. It is likely that the special case of Theorem 1.1 concerning locally finite right-angled buildings could also be deduced from [Laz], using the fact that right-angled buildings can be cubulated.

Notice that a building whose type-preserving automorphism group is chamber-transitive, is necessarily semi-regular. The following is thus immediate from Theorem 1.1.

Corollary 1.2. *Let X be an irreducible thick right-angled building of non-spherical type. If* $\mathrm{Aut}(X)^+$ *is chamber-transitive, then it is strongly transitive and abstractly simple.*

In the special case when X is **locally finite**, i.e. when $q_i < \infty$ for all $i \in I$, the group G endowed with the compact-open topology is a second countable totally disconnected locally compact group. It is compactly generated since it acts chamber-transitively on X. In particular Theorem 1.1 provides a large family of compactly generated simple locally compact groups. Our next goal is to describe their rich local structure.

A general study of the local structure of compactly generated, topologically simple, totally disconnected locally compact groups is initiated in [CRW12]. The main objects of consideration in that study are the **locally normal subgroups**, namely the compact subgroups whose normaliser is open. The trivial subgroup, as well as the compact open subgroups, are obviously locally normal, considered as trivial. It is important to observe that a compactly generated, locally compact group can be topologically simple and nevertheless possess non-trivial locally normal subgroups. Basic examples of such groups are provided by the type-preserving automorphism group of semi-regular locally finite tree. It turns out that the group of type-preserving automorphisms of an arbitrary semi-regular locally finite tree always admits non-trivial locally normal subgroups; some of them even split non-trivially as direct products (see Lemma 9.1 below). The case of trees has however a special additional property: some compact *open* subgroups split as a direct product of infinite closed subgroups; the corresponding factors are *a fortiori* locally normal and non-trivial. It is thus natural to ask for which right-angled buildings that situation occurs, beyond the case of trees. The following provides a complete answer to this question, implying in particular that open subgroups admit non-trivial product decompositions only under very special circumstances.

Theorem 1.3. *Let X be a building of right-angled type (W,I) and prescribed thicknesses $(q_i)_{i\in I}$, with $2 < q_i < \infty$ for all $i \in I$. Assume that (W,I) is irreducible non-spherical.*

Then the following assertions are equivalent:

(i) *All open subgroups of* $G = \mathrm{Aut}(X)^+$ *are indecomposable.*
(ii) G *is one-ended.*
(iii) W *is one-ended.*

By **indecomposable**, we mean the non-existence of a non-trivial direct product decomposition. The **set of ends** of a compactly generated locally compact group is defined with respect to compact generating sets in the same way as for discrete groups (see [Abe74]). Notice that Theorem 1.3 establishes a relation between the *local* structure of G (because the existence of an open subgroup splitting non-trivially as a product can be detected in arbitrarily small identity neighbourhoods) and its *asymptotic* properties.

The condition that W is one-ended can easily be read on the Coxeter diagram (see Theorem 9.2 for a precise formulation).

It follows from Theorem 1.3 that, if X is a Bourdon building, then compact open subgroups of $\mathrm{Aut}(X)^+$ are indecomposable, but they have normal subgroups that split non-trivially as products. With Theorem 1.3 at hand, one can construct buildings X of arbitrarily large dimension whose automorphism group has that property. In fact, one can arrange that compact open subgroups are indecomposable, but possess a normal series of arbitrarily large length all of whose subquotients split non-trivially as products.

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2. Projections and parallel residues

A fundamental feature of buildings is the existence of combinatorial projections between residues. We briefly recall their basic properties, which will be frequently used in the sequel. All the properties which we do not prove in detail are established in [Tit74, §3.19].

Let X be a building. Given a chamber $c \in \mathrm{Ch}(X)$ and a residue σ in X, the **projection** of c on σ is the unique chamber of $\mathrm{Ch}(\sigma)$ that is closest to c. It is denoted by $\mathrm{proj}_\sigma(c)$. For any chamber $d \in \mathrm{Ch}(\sigma)$, there is a minimal gallery from c to d passing through $\mathrm{proj}_\sigma(c)$. Moreover, any apartment containing c and meeting $\mathrm{Ch}(\sigma)$ also contains $\mathrm{proj}_\sigma(c)$. An important property of proj is that it does not increase the numerical distance between chambers: for all $c, c' \in \mathrm{Ch}(X)$, the numerical distance from $\mathrm{proj}_\sigma(c)$ to $\mathrm{proj}_\sigma(c')$ is bounded above by the numerical distance from c to c'.

If σ and τ are two residues, then the set

$$\{\mathrm{proj}_\sigma(c) \mid c \in \mathrm{Ch}(\tau)\}$$

is the chamber-set of a residue contained in σ. That residue is denoted by $\mathrm{proj}_\sigma(\tau)$. The rank of $\mathrm{proj}_\sigma(\tau)$ is bounded above by the ranks of both σ and τ.

We shall often use the following crucial property of the projection map; we outsource its statement for the ease of reference.

Lemma 2.1. *Let* R, S *be two residues such that* $\mathrm{Ch}(R) \subset \mathrm{Ch}(S)$. *Then for any residue* σ, *we have* $\mathrm{proj}_R(\sigma) = \mathrm{proj}_R(\mathrm{proj}_S(\sigma))$.

Proof. See [Tit74, 3.19.5]. □

Two residues σ and τ are called **parallel** if $\mathrm{proj}_\sigma(\tau) = \sigma$ and $\mathrm{proj}_\tau(\sigma) = \tau$. In that case, the chamber sets of σ and τ are mutually in bijection under the respective

projection maps. Since the projection map between residues does not increase the rank, it follows that two parallel residues have the same rank. A basic examples of parallel residues is provided by two opposite residues in a spherical building.

The following result shows that two residues are parallel if and only if they share the same set of walls in every apartment containing them both. This useful criterion allows one to detect parallelism of residues by just looking at parallelism among panels.

Lemma 2.2. *Let R and R' be two residues. Then R and R' are parallel if and only if for all panels σ of R and σ' of R', the projections $\mathrm{proj}_{R'}(\sigma)$ and $\mathrm{proj}_R(\sigma')$ are both panels.*

Proof. The 'only if' part is clear from the definition. Assume that R and R' are not parallel. Up to swapping the roles of R and R', we may thus assume that $\mathrm{proj}_R(R')$ is a proper residue of R. Let then c and d be a pair of adjacent chambers in R so that c is the projection of some chamber of R' and d is not. Then $c' = \mathrm{proj}_{R'}(c)$ is adjacent to $d' = \mathrm{proj}_{R'}(d)$. If the latter two chambers coincide, then the projection on R' of the panel shared by c and d is a chamber and not a panel, and the desired condition holds. Otherwise the panel shared by c and d is parallel to the panel shared by c' and d'. This implies that $c = \mathrm{proj}_R(c')$ and $d = \mathrm{proj}_R(d')$, contradicting that d does not belong to the chamber-set of $\mathrm{proj}_R(R')$. □

Another useful fact is the following.

Lemma 2.3. *Let R and R' be two residues.*
Then $\mathrm{proj}_{R'}(R)$ and $\mathrm{proj}_R(R')$ are parallel.

Proof. Let σ be a panel contained in $\mathrm{proj}_R(R')$. Then there is a panel σ' in R' such that $\sigma = \mathrm{proj}_R(\sigma')$. It follows that σ and σ' are parallel. Therefore, we have

$$\sigma' = \mathrm{proj}_{\sigma'}(\sigma) = \mathrm{proj}_{\sigma'}(\mathrm{proj}_{R'}(\sigma)),$$

where the second equality follows from Lemma 2.1. It follows that $\mathrm{proj}_{R'}(\sigma)$ is a panel. Clearly, we have $\mathrm{proj}_{R'}(\sigma) = \mathrm{proj}_{\mathrm{proj}_{R'}(R)}(\sigma)$. This shows that the projection of σ to $\mathrm{proj}_{R'}(R)$ is a panel.

By symmetry, the projection of any panel of $\mathrm{proj}_{R'}(R)$ to $\mathrm{proj}_R(R')$ is also a panel. By Lemma 2.2, we infer that $\mathrm{proj}_{R'}(R)$ and $\mathrm{proj}_R(R')$ are parallel. □

Corollary 2.4. *Let σ and σ' be panels.*
If two chambers of σ' have distinct projections on σ, then σ and σ' are parallel.

Proof. If two chambers of σ' have distinct projections on σ, then $\mathrm{proj}_\sigma(\sigma')$ is a panel, which is thus the whole of σ. Therefore $\mathrm{proj}_{\sigma'}(\sigma)$ cannot be reduced to a single chamber by Lemma 2.3. The result follows. □

Corollary 2.4 indicates how straightforward it is to check that two panels are parallel. It will be used frequently.

We shall see that parallelism of residues has a very special behaviour in right-angled buildings. For instance, we have the following useful criterion.

Proposition 2.5. *Let X be a right-angled building of type (W, I).*

(i) Two parallel residues have the same type.
(ii) Given a residue R of type J, a residue R' is parallel to R if and only if R' is of type J and R and R' are both contained in a residue of type $J \cup J^\perp$.

We recall that $J^\perp$ is the subset of I defined by

$$J^\perp = \{i \in I \mid i \notin J,\ ij = ji \text{ for all } j \in J\}.$$

Proof of Proposition 2.5. (i) In a right-angled building, any two panels lying on a common wall in some apartment have the same type. That two parallel residues have the same type is thus a consequence of Lemma 2.2.

(ii) Any two residues of type J in a building of type $J \cup J^\perp$ are parallel. This implies that the 'if' part holds.

Assume now that R and R' are parallel. Let $c \in \mathrm{Ch}(R)$ and $c' = \mathrm{proj}_{R'}(c)$. We show by induction on $d(c, C')$ that the type of every panel crossed by a minimal gallery from c to c' belongs to J'. Let $c = d_0, d_1, \ldots, d_n = C'$ be such a minimal gallery. Let also i be the type of the panel shared by $c = d_0$ and d_1, and let σ_i denote that i-panel. For any $j \in J$, let also σ_j be the j-panel of c. By Lemma 2.2, the projection $\sigma'_j = \mathrm{proj}_{R'}(\sigma_j)$ is a panel. The panels σ_j and σ'_j lie therefore on a common wall in any apartment containing them both. If i and j did not commute, then the wall $\mathcal{W}_i$ containing the panel σ_i in such an apartment would be disjoint from the wall $\mathcal{W}_j$ containing σ_j. This implies c' is separated from $\mathcal{W}_j$ by the wall $\mathcal{W}_i$, which prevents the panel σ'_j from lying on $\mathcal{W}_j$. This shows that $ij = ji$. In other words, we have $i \in J^\perp$.

Let next R_1 be the J-residue containing d_1, and let S be the $(J \cup \{i\})$-residue containing c. Thus R and R_1 are both contained in S.

We claim that R_1 is parallel to R'. In order to establish the claim, we first notice that $\mathrm{proj}_{R'}(S) = R'$, since $R \subset S$. By Lemma 2.3, the residue $R' = \mathrm{proj}_{R'}(S)$ is parallel to $\mathrm{proj}_S(R')$. In particular $\mathrm{proj}_S(R')$ is of type J by Assertion (i). Since $i \in J^\perp$, all J-residues in S contain exactly one chamber of σ_i. Since σ_i is not contained in $\mathrm{proj}_S(R')$, it follows that all chambers of R' have the same projection on σ_i; that projection is the unique chamber of $\mathrm{Ch}(\sigma_i) \cap \mathrm{Ch}(\mathrm{proj}_S(R'))$. By construction, we have $\mathrm{proj}_{\sigma_i}(c') = d_1$; we deduce that d_1 belongs to $\mathrm{Ch}(\mathrm{proj}_S(R'))$. This proves that $\mathrm{proj}_S(R')$ is the J-residue of d_1; it coincides therefore with R_1.

Thus we have shown that $R' = \mathrm{proj}_{R'}(S)$ and that $R_1 = \mathrm{proj}_S(R')$, and those residues are parallel by Lemma 2.3. The claim stands proven.

The claim implies by induction on n that R_1 and R' are contained in a common residue of type $J \cup J^\perp$. That residue must also contain R, since R and R_1 are contained in a common residue of type $J \cup \{i\} \subseteq J \cup J^\perp$. This finishes the proof. □

Corollary 2.6. *Let X be a right-angled building.*

Then parallelism of residues is an equivalence relation.

Proof. This follows from Proposition 2.5(ii). □

We emphasize that parallelism of panels is *not an equivalence relation* in general. In fact, we have the following characterization of right-angled buildings.

Proposition 2.7. *Let X be a thick building.*

Then parallelism of residues is an equivalence relation if and only if X is right-angled.

Proof. By Corollary 2.6, it suffices to show that if X is not right-angled, then parallelism of panels is not an equivalence relation. If X is not right-angled, then it contains a residue R which is an irreducible generalized polygon. Let σ and σ' be two distinct panels of the same type in R, at minimal distance from on another. It follows that σ and σ' are not opposite in R, and thus not parallel since they do not

lie on a common wall in apartments containing σ and σ'. By [Tit74, 3.30], there is a panel τ in R which is opposite both σ and σ'. Thus σ is parallel to τ and τ is parallel to σ'. Parallelism is thus not a transitive relation. □

3. Wall-residues and wings

Let X be a right-angled building of type (W, I).

Since parallelism of residues is an equivalence relation by Corollary 2.6, it is natural to ask what the equivalence classes are. The answer is in fact already provided by Proposition 2.5: the parallel classes of J-residues are the sets of J-residues contained in a common residue of type $J \cup J^\perp$.

Given a residue R of type J, we will denote the unique residue of type $J \cup J^\perp$ containing R by $\bar{R}$. The special case of panels is the most important one. A residue of the form $\bar{\sigma}$, with σ a panel, will be called a **wall-residue**.

In the case when (W, I) is a right-angled Fuchsian Coxeter group, wall-residues are what Marc Bourdon calls **wall-trees**, see [Bou97]. The terminology is motivated by the following observation: if the intersection of a wall-residue with an apartment is non-empty, then it is a wall in that apartment.

Our next step is to show how residues determine a partition of the chamber-set of the ambient building into convex pieces. To this end, we need some additional terminology and notation.

To any $c \in \mathrm{Ch}(X)$ and $J \subset I$, we associate the set

$$X_J(c) = \{x \in \mathrm{Ch}(X) \mid \mathrm{proj}_\sigma(x) = c\},$$

where $\sigma = \mathrm{Res}_J(c)$ is the J-residue of the chamber c. We call $X_J(c)$ the **J-wing** containing c. If $J = \{i\}$ is a singleton, we write $X_i(c)$ and call it the **i-wing** of c. A **wing** is a J-wing for some $J \subseteq I$.

Proposition 3.1. *In a right-angled building, wings are convex.*

Proof. Fix $c \in \mathrm{Ch}(X)$ and $J \subseteq I$.

We claim that

$$X_J(c) = \bigcap_{i \in J} X_i(c).$$

The inclusion $\subseteq$ is clear. To check the reverse inclusion, let x be a chamber whose projection onto $R = \mathrm{Res}_J(c)$ is different from c. Then there is a minimal gallery from x to c via $x' = \mathrm{proj}_R(x)$. Let i be type of the last panel crossed by that gallery, and let σ be that panel. By construction we have $\mathrm{proj}_\sigma(x) \neq c$. Moreover, since $x' \neq c$, we have $i \in J$. This implies that $x \notin X_i(c)$. The claim stands proven.

In view of the claim, it suffices to prove that a wing of the form $X_i(c)$ with $i \in I$ is convex. Let σ be the i-panel of c. Let also $d, d' \in X_i(c)$ and let $d = d_0, d_1, \dots, d_n = d'$ be a minimal gallery joining them.

Assume that the gallery is not entirely contained in $X_i(c)$. Let j be the minimal index such that $d_{j+1} \notin X_i(c)$, and let j' be the maximal index such that $d_{j'-1} \notin X_i(c)$. Thus $j' > j$.

By construction, the panel σ_j shared by d_j and d_{j+1} is parallel to σ. Similarly, so is the panel $\sigma_{j'}$ shared by $d_{j'}$ and $d_{j'-1}$. Therefore, by Proposition 2.5, the set $\mathrm{Ch}(\sigma) \cup \mathrm{Ch}(\sigma_j) \cup \mathrm{Ch}(\sigma_{j'})$ is contained in $\mathrm{Ch}(\bar{\sigma})$ (where as above $\bar{\sigma}$ denotes that $(\{i\} \cup \{i\}^\perp)$-residue containing σ).

Setting $d'_k = \mathrm{proj}_{\bar\sigma}(d_k)$, we infer that the sequence

$$d_j = d'_j = d'_{j+1}, d'_{j+2}, \dots, d'_{j'-2}, d'_{j'-1} = d'_{j'} = d_{j'}$$

is a gallery that is strictly shorter than the given minimal gallery $d_j, d_{j+1}, \dots, d_{j'}$. This is absurd. □

By definition of the projection, the set $\mathrm{Ch}(X)$ is the disjoint union of the wings $X_J(d)$ over all $d \in \mathrm{Ch}(\mathrm{Res}_J(c))$. It thus follows from Proposition 3.1 that any residue containing q chambers yields a partition of the building into q convex subsets.

For the sake of future references, we record the following fact.

Lemma 3.2. *Let $i \in I$, let $c \in \mathrm{Ch}(X)$ and let $\sigma = \mathrm{Res}_i(c)$.*

For any $x \in X_i(c)$ and $x' \notin X_i(c)$, the gallery from x to x' obtained by concatenating a minimal gallery from x to $\mathrm{proj}_{\bar\sigma}(x)$, a minimal gallery from $\mathrm{proj}_{\bar\sigma}(x)$ to $\mathrm{proj}_{\bar\sigma}(x')$, and a minimal gallery from $\mathrm{proj}_{\bar\sigma}(x')$ to x', is minimal.

Proof. A gallery is minimal if and only if its length equals the numerical distance between its extremities. Therefore, it suffices to show that there is some minimal gallery from x to x' passing through $\mathrm{proj}_{\bar\sigma}(x)$ and $\mathrm{proj}_{\bar\sigma}(x')$.

Let $\gamma = (x = x_0, x_1, \dots, x_n = x')$ be some minimal gallery from x to x'. Since $x' \notin X_i(c)$, the gallery γ must cross some panel which is parallel to σ. By Proposition 2.5, this implies that the gallery γ meets the residue $\bar\sigma$.

Let j (resp. j') be the minimal (resp. maximal) index k such that the chamber x_k of γ belongs to $\mathrm{Ch}(\bar\sigma)$. Then there is a minimal gallery γ_j from x to x_j (resp. $\gamma_{j'}$ from $x_{j'}$ to x') passing through $\mathrm{proj}_{\bar\sigma}(x)$ (resp. $\mathrm{proj}_{\bar\sigma}(x')$). By concatenating γ_j and $\gamma_{j'}$ with the gallery $x_j, x_{j+1}, \dots, x_{j'}$, we obtain a gallery $\tilde\gamma$, of the same length as γ, and joining x to x'. Thus $\tilde\gamma$ is minimal. By construction, it passes through $\mathrm{proj}_{\bar\sigma}(x)$ and $\mathrm{proj}_{\bar\sigma}(x')$. □

Crucial to our purposes is the relation of inclusion between wings described by the following.

Lemma 3.3. *Let $i, i' \in I$ and $c, c' \in \mathrm{Ch}(X)$. If $i \neq i'$, then we assume in addition that $m_{ii'} = \infty$.*

If $c \in X_{i'}(c')$ and $c' \notin X_i(c)$, then $X_i(c) \subset X_{i'}(c')$.

Proof. Let σ (resp. σ') be the i-panel (resp. i'-panel) of c (resp. c'). Let $d \in \mathrm{Ch}(X)$ with $\mathrm{proj}_\sigma(d) = c$. We need to show that then $\mathrm{proj}_{\sigma'}(d) = c'$.

Let $\bar\sigma = \mathrm{Res}_{i \cup i^\perp}(c)$. Let $x = \mathrm{proj}_{\bar\sigma}(c')$ and $y = \mathrm{proj}_{\bar\sigma}(d)$.

Let $c'' \in \mathrm{Ch}(\sigma')$ be different from c'. We need to show that $\mathrm{dist}(d, c'') = \mathrm{dist}(d, c') + 1$.

In order to see this, we first claim that $\mathrm{proj}_{\bar\sigma}(c'') = x$. If this were not the case, then $\mathrm{proj}_{\bar\sigma}(\sigma')$ would be parallel to σ'; it would thus coincide with the i'-panel of x by Proposition 2.5(i). We denote the i'-panel of x by τ. The inclusion of τ in $\bar\sigma$ implies that i' belongs to $i \cup i^\perp$. By hypothesis, this forces $i = i'$. It now follows from Proposition 2.5(ii) that τ is parallel to σ. Hence σ and σ' are parallel by Corollary 2.6. Since $\mathrm{proj}_\sigma(c') \neq c$, it then follows that $\mathrm{proj}_{\sigma'}(c) \neq c'$, whence $c' \notin X_i(c)$, a contradiction. This proves that $\mathrm{proj}_{\bar\sigma}(c'') = x$ as claimed.

By assumption, we have $d \in X_i(c)$ and $c' \notin X_i(c)$. Therefore, Lemma 3.2 implies that

$$\mathrm{dist}(d, c') = \mathrm{dist}(d, y) + \mathrm{dist}(y, x) + \mathrm{dist}(x, c').$$

Moreover, by the claim, we also have

$$\operatorname{dist}(d, c'') = \operatorname{dist}(d, y) + \operatorname{dist}(y, x) + \operatorname{dist}(x, c'').$$

So it suffices to show that $\operatorname{dist}(x, c'') = \operatorname{dist}(x, c') + 1$. But Lemma 3.2 applied to c and c'' also implies that

$$\operatorname{dist}(c, c'') = \operatorname{dist}(c, c') + 1 = \operatorname{dist}(c, x) + \operatorname{dist}(x, c') + 1,$$

whence $\operatorname{dist}(x, c'') = \operatorname{dist}(x, c') + 1$, as desired. □

We shall also need the following additional properties of the subsets $X_i(c)$.

Lemma 3.4. *Let $i \in I$, let $c, x \in \operatorname{Ch}(X)$ and let σ be the i-panel of c and $\bar\sigma$ be the residue of type $i \cup i^\perp$ containing c.*

Assume that $\operatorname{proj}_{\bar\sigma}(x) = c$ and let $n = \operatorname{dist}(x, c)$. Then the ball $B(x, n+1) \subset \operatorname{Ch}(X)$ of radius $n+1$ around x is entirely contained in $X_i(c) \cup \operatorname{Ch}(\sigma)$.

Proof. Choose a chamber $y \in B(x, n+1) - X_i(c)$. Let $x = x_0, x_1, \dots, x_{n+1} = y$ be a minimal gallery. Since $\operatorname{proj}_{\bar\sigma}(x) = c$, we have $\operatorname{proj}_\sigma(x) = c$, whence $x \in X_i(c)$. Let $k_0 = \min\{\ell \mid x_\ell \notin X_i(c)\}$. Thus $k_0 > 0$ and $x_s \in X_i(c)$ for all $s \in \{0, \dots, k_0 - 1\}$.

We next observe that the panel σ' shared by x_{k_0-1} and x_{k_0} is parallel to σ by Corollary 2.4. Hence x_{k_0-1} and x_{k_0} both belong to $\operatorname{Ch}(\bar\sigma)$ by Proposition 2.5(ii). In particular we have

$$k_0 - 1 = \operatorname{dist}(x, x_{k_0-1}) \geq \operatorname{dist}(x, \operatorname{proj}_{\bar\sigma}(x)) = n.$$

Therefore $k_0 = n+1$ and x_{k_0-1} must coincide with the projection $\operatorname{proj}_{\bar\sigma}(x) = c$. This yields $\sigma' = \sigma$ and $y \in \operatorname{Ch}(\sigma)$. □

Lemma 3.5. *Let $J \subset I$ and $i \in I - J$. Given a J-residue R and a chamber $c \in \operatorname{Ch}(R)$, we have $\operatorname{Ch}(R) \subseteq X_i(c)$.*

Proof. Let $c \in \operatorname{Ch}(R)$ and σ be the i-panel of c. Suppose for a contradiction that $\operatorname{proj}_\sigma(R) = \sigma$. Then $\operatorname{proj}_R(\sigma) = \tau$ is panel which is parallel to σ by Lemma 2.3. Therefore σ and τ are both i-panels by Proposition 2.5(i). Since τ is contained in R, this contradicts that $i \notin J$. □

4. Extending local automorphisms

The following important result was shown by Haglund–Paulin.

Proposition 4.1 (Haglund–Paulin)**.** *Let X be a semi-regular right-angled building. For any residue R of X and any $\alpha \in \operatorname{Aut}(R)^+$, there is $\tilde\alpha \in \operatorname{Aut}(X)^+$ stabilising R and such that $\tilde\alpha|_{\operatorname{Ch}(R)} = \alpha$.*

Proof. See Proposition 5.1 in [HP03]. □

In other words, this means that the canonical map from the stabiliser $\operatorname{Stab}_{\operatorname{Aut}(X)^+}(R)$ to $\operatorname{Aut}(R)^+$ is surjective.

It will be important to our purposes to ensure that the extension constructed in Proposition 4.1 can be chosen satisfy some additional constraints. In particular, we record the following.

Proposition 4.2. *Let X be a semi-regular right-angled building of type (W, I). Let $i \in I$ and σ be an i-panel.*

Given any permutation $\alpha \in \operatorname{Sym}(\operatorname{Ch}(\sigma))$, there is $\tilde\alpha \in \operatorname{Aut}(X)^+$ stabilising σ satisfying the following two conditions:

(i) $\tilde\alpha|_{\operatorname{Ch}(\sigma)} = \alpha$;

(ii) $\tilde{\alpha}$ fixes all chambers of X whose projection to σ is fixed by α.

Proof. Let $\sigma^{\perp}$ denote a residue of type $\{i\}^{\perp}$ containing some chamber of σ. The residue $\bar{\sigma}$ then splits as a direct product $\bar{\sigma} = \sigma \times \sigma^{\perp}$. We define $\beta \in \mathrm{Aut}(\bar{\sigma})^{+}$ as $\beta = \alpha \times \mathrm{Id}$. By Proposition 4.1, the automorphism β of $\bar{\sigma}$ extends to some (type-preserving) automorphism $\tilde{\beta}$ of X.

We now define a map $\tilde{\alpha} \colon \mathrm{Ch}(X) \to \mathrm{Ch}(X)$ as follows: for each $c \in \mathrm{Ch}(X)$, we set

$$\tilde{\alpha}(c) = \begin{cases} c & \text{if } \alpha(\mathrm{proj}_{\sigma}(c)) = \mathrm{proj}_{\sigma}(c); \\ \tilde{\beta}(c) & \text{otherwise.} \end{cases}$$

Clearly the map $\tilde{\alpha}$ satisfies the desired condition (ii). Moreover, we have $\tilde{\alpha}|_{\mathrm{Ch}(\bar{\sigma})} = \beta$, from which it follows that condition (i) holds as well.

It remains to check that $\tilde{\alpha}$ is an automorphism. To this end, let x and y be any two chambers and denote by x' and y' their projections on σ.

If $x' = y'$, then we have either $(\tilde{\alpha}(x), \tilde{\alpha}(y)) = (x, y)$, or $(\tilde{\alpha}(x), \tilde{\alpha}(y)) = (\tilde{\beta}(x), \tilde{\beta}(y))$. In both cases, it follows that $\tilde{\alpha}$ preserves the Weyl-distance from x to y.

Assume now that $x' \neq y'$. Let then x'' and y'' denote the projections of x and y on $\bar{\sigma}$. By Lemma 3.2, it suffices to show that $\tilde{\alpha}$ preserves the Weyl-distance from x to x'', the Weyl-distance from x'' to y'' and the Weyl distance from y'' to y. Since wings are convex by Proposition 3.1, and since the restriction of $\tilde{\alpha}$ on each wing of σ preserves the Weyl-distance, it follows that $\tilde{\alpha}$ preserves the Weyl-distance from x to $x''$and from y'' to y. That the Weyl-distance from x'' to y'' is preserved is clear since the restriction of $\tilde{\alpha}$ to $\mathrm{Ch}(\bar{\sigma})$ is the automorphism β.

This proves that $\tilde{\alpha}$ preserves the Weyl-distance from x to y. Thus $\tilde{\alpha}$ is an automorphism. □

5. Fixators of wings

As before, let X be a right-angled building of type (W, I).

The subsets $X_i(c)$ are analogues of half-trees in the case W is infinite dihedral. In view of this analogy, we shall consider the subgroups of $\mathrm{Aut}(X)^{+}$ denoted by $V_i(c)$ and $U_i(c)$, consisting respectively of those automorphisms supported on $X_i(c)$ and on its complement. In symbols, this yields

$$U_i(c) = \{g \in \mathrm{Aut}(X)^{+} \mid g(x) = x \text{ for all } x \in X_i(c)\},$$

and

$$V_i(c) = \{g \in \mathrm{Aut}(X)^{+} \mid g(x) = x \text{ for all } x \notin X_i(c)\}.$$

Clearly $U_i(c)$ and $V_i(c)$ both fix c and stabilise σ. Moreover they commute and have trivial intersection, since their support are disjoint. The following implies that they are both non-trivial.

Lemma 5.1. *Assume that X is thick and semi-regular. Let $i, j \in I$ be such that $m_{i,j} = \infty$. Then for all $c \in \mathrm{Ch}(X)$, the groups $U_i(c)$ and $V_i(c)$ are non-abelian.*

Proof. Let $x \neq c$ be a chamber j-adjacent to c. Then $X_j(x) \subset X_i(c)$ by Lemma 3.3. This implies that $U_i(c)$ fixes pointwise $X_j(x)$ for all chambers $x \neq c$ that are j-adjacent to but different from c. In particular $U_i(c)$ is contained in $V_j(c)$, so it suffices to show that $U_i(c)$ is non-abelian.

Proposition 4.2 implies that $U_i(c)$ is non-trivial; so are in particular $V_i(x)$ and $V_j(x)$ for all $x \in \mathrm{Ch}(X)$ in view of what we have just observed.

For each $c' \neq c$ be a chamber that is i-adjacent to c, the group $V_i(c')$ is contained in $U_i(c)$. Moreover, if c', c'' are two distinct such chambers, the groups $V_i(c')$ and $V_i(c'')$

are different since they are non-trivial and have disjoint support. By Proposition 4.2, there is $u \in U_i(c)$ mapping c' to c''. We then have $uV_i(c')u^{-1} = V_i(c'') \neq V_i(c')$. In particular u does not commute with $V_i(c')$, which proves that $U_i(c)$ is not abelian. □

Given $G \leq \mathrm{Aut}(X)$, the pointwise stabiliser of the chamber set $\mathrm{Ch}(R)$ of a residue R is denoted by $\mathrm{Fix}_G(R)$. We shall next describe how the groups $U_i(c)$ and $V_i(c)$ provide convenient generating sets for the pointwise stabilisers of residues and balls in X. We start with wall-residues.

Proposition 5.2. *Let X be a thick semi-regular right-angled building of type (W, I). Let $c \in \mathrm{Ch}(X)$ and $i \in I$, and let $R = \mathrm{Res}_{i\cup i^\perp}(c)$ be the residue of type $i \cup i^\perp$ of c.*

Then we have

$$\mathrm{Fix}_{\mathrm{Aut}(X)^+}(R) = \prod_{d \sim_i c} V_i(d).$$

Proof. By Lemma 3.2, each element of $\mathrm{Fix}_{\mathrm{Aut}(X)^+}(R)$ stabilises $X_i(d)$ for all $d \sim_i c$. Thus there is a canonical injective homomorphism

$$\mathrm{Fix}_{\mathrm{Aut}(X)^+}(R) \to \prod_{d \sim_i c} V_i(d).$$

In order to show that it is surjective, it suffices to show that, given $g \in \mathrm{Fix}_{\mathrm{Aut}(X)^+}(R)$, the permutation $\tilde{g}$ of the chamber set defined by

$$\tilde{g}\colon \mathrm{Ch}(X) \to \mathrm{Ch}(X) \colon x \mapsto \begin{cases} g(x) & \text{if } x \in X_i(c) \\ x & \text{otherwise} \end{cases}$$

is an autorphism of X. To see this, let $x, y \in \mathrm{Ch}(X)$ and let $\delta\colon \mathrm{Ch}(X)\times\mathrm{Ch}(X) \to W$ denote the Weyl distance. We need to show that $\delta(\tilde{g}(x), \tilde{g}(y)) = \delta(x, y)$. By the definition of $\tilde{g}$, it suffices to consider the case when $x \in X_i(c)$ and $y \notin X_i(c)$ (or vice-versa). By Lemma 3.2, we have

$$\delta(x, y) = \delta(x, x')\delta(x', y')\delta(y', y),$$

where $x' = \mathrm{proj}_R(x)$, $y' = \mathrm{proj}_R(y)$ and $R = \mathrm{Res}_{i\cup i^\perp}(c)$. Moreover, the element $g \in \mathrm{Aut}(X)^+$ fixes x', y and y' and preserves R and $X_i(c)$. Thus we have $\mathrm{proj}_R(\tilde{g}(x)) = \mathrm{proj}_R(g(x)) = x'$ and, invoking Lemma 3.2 once more, we deduce

$$\begin{aligned} \delta(\tilde{g}(x), \tilde{g}(y)) &= \delta(g(x), y) \\ &= \delta(g(x), x')\delta(x', y')\delta(y', y) \\ &= \delta(x, x')\delta(x', y')\delta(y', y) \\ &= \delta(x, y) \end{aligned}$$

as desired. □

6. Strong transitivity

Proposition 6.1. *Let X be a thick semi-regular right-angled building.*

Then the group $\mathrm{Aut}(X)^+$ is strongly transitive on X.

We need the following basic consequence of Proposition 4.2.

Lemma 6.2. *Let X be a thick semi-regular right-angled building of type (W, I).*

Let $x \in \mathrm{Ch}(X)$ and $n \geq 0$. Let $c_1, \dots, c_t \in \mathrm{Ch}(X)$ be at distance n from x, and let $i_1, \dots, i_t \in I$ be such that $\mathrm{proj}_{\bar{\sigma}_s}(x) = c_s$, where $\bar{\sigma}_s = \mathrm{Res}_{i_s\cup i_s^\perp}(c_s)$ for all $s = 1, \dots, t$.

For all $s \in \{1, \dots, t\}$, let π_s be an arbitrary permutation of $\mathrm{Ch}(\sigma_s)$ fixing c_s, where σ_s is the i_s-panel of c_s. Then there is $g \in \langle U_{i_s}(c_s) \mid s = 1, \dots, t\rangle$ whose restriction to $\mathrm{Ch}(\sigma_s)$ is π_s for all s. Moreover g fixes pointwise $B(x, n+1) - \bigcup_{s=1}^t \mathrm{Ch}(\sigma_s)$.

Proof. By Lemma 3.4, every element of $U_{i_s}(c_s)$ fixes pointwise $B(x, n+1) - \mathrm{Ch}(\sigma_s)$. Therefore, if we find $g_s \in U_{i_s}(c_s)$ with $g_s(d_s) = d'_s$, then the product $g = g_1 \dots g_t$ will enjoy the desired properties. The existence of $g_s \in U_{i_s}(c_s)$ with $g|_{\mathrm{Ch}(\sigma_s)} = \pi_s$ is immediate from Proposition 4.2. □

Proof of Proposition 6.1. As observed by Haglund–Paulin [HP03], Proposition 4.1 readily implies that $\mathrm{Aut}(X)^+$ is chamber transitive. We need to show that given a chamber $c \in \mathrm{Ch}(X)$ and two apartments A, A' containing c, there is an element $g \in \mathrm{Aut}(X)^+$ fixing c and mapping A to A'.

Set $g_0 = \mathrm{Id}$ and let $n > 0$. We shall construct by induction on n an element $g_n \in \mathrm{Aut}(X)^+$ with the following properties:

- g_n fixes pointwise the ball of radius $n-1$ around c;
- $g_n g_{n-1} \dots g_0(A) \cap A' \supseteq B(c, n) \cap A'$, where $B(c, n)$ is the ball of radius n around c.

The first property ensures that the sequence $(g_n g_{n-1} \dots g_0)_{n \geq 0}$ pointwise converges to a well defined automorphism $g_\infty \in \mathrm{Aut}(X)^+$. The second property yields $g_\infty(A) = A'$, as desired.

Let $n \geq 0$, and suppose that $g_0, g_1, \dots, g_n$ have already been constructed. Set $A_n = g_n g_{n-1} \dots g_0(A)$. Thus $A_n \cap A'$ contains every chamber of A' at distance at most n from c.

We need to construct an automorphism $g_{n+1} \in \mathrm{Aut}(X)^+$ fixing $B(c, n)$ pointwise and such that $g_{n+1}(A_n) \cap A'$ contains $B(c, n+1) \cap A'$.

Let E be the set of those chambers in $B(c, n+1) \cap A'$ that are not contained in A_n. Notice that E is finite (since $B(c, n+1) \cap A'$ is so).

If E is empty, then we set $g_{n+1} = \mathrm{Id}$ and we are done. Otherwise we enumerate $E = \{x'_1, \dots, x'_t\}$ and consider $s \in \{1, \dots, t\}$. Let y_s be the first chamber different from x'_s on a minimal from x'_s to c. Thus $y_s \in B(c, n) \cap A'$, and hence $y_s \in A_n$. Let σ_s be the panel shared by x'_s and y_s, let $i_s \in I$ be its type and let $x_s \in A_n$ be the unique chamber which is i_s-adjacent to, but different from, y_s. By Lemma 6.2, there is an element $g_{n+1} \in \langle U_{i_s}(y_s) \mid s = 1, \dots, t\rangle$ which maps x_s to x'_s for all s, and fixes $B(c, n)$ pointwise. Thus g_{n+1} has the requested properties, and we are done. □

We are thus in a position to invoke Tits' transitivity lemma:

Corollary 6.3. *Let X be a thick semi-regular right-angled building of irreducible type.*

Then every non-trivial normal subgroup of $\mathrm{Aut}(X)^+$ *is transitive on* $\mathrm{Ch}(X)$.

Proof. Since $\mathrm{Aut}(X)^+$ is strongly transitive by Proposition 6.1, this follows from Proposition 2.5 in [Tit64]. □

In case when X is locally finite, the strong transitivity guaranteed by Proposition 6.1 is already enough to ensure that the intersection of all non-trivial closed normal subgroups of $\mathrm{Aut}(X)^+$ is non-trivial, topologically simple and cocompact, see [CM11, Corollary 3.1]. This is of course a much weaker conclusion than Theorem 1.1.

We record the following consequence of Proposition 5.2 and Corollary 6.3 which is crucial for Theorem 1.1. The proof is based on Tits' commutator lemma (Lemma 4.3 in [Tit70] or Lemma 6.2 in [HP98]).

Lemma 6.4. *Assume that the Coxeter system (W, I) is irreducible and that X is thick.*

Then for any wall-residue R, every non-trivial normal subgroup of $\mathrm{Aut}(X)^+$ *contains* $\mathrm{Fix}_{\mathrm{Aut}(X)^+}(R)$.

Proof. Let $N < \mathrm{Aut}(X)^+$ be a non-trivial normal subgroup.

Let $i \in I$, $c_0 \in \mathrm{Ch}(X)$ and $R = \mathrm{Res}_{i \cup i^\perp}(c_0)$. In view of the product decomposition provided by Proposition 5.2, it suffices to show that the group $V_i(x)$ is contained in N for each $x \sim_i c_0$.

Fix thus $x \sim_i c_0$. Since the building X is thick, there is a chamber $c_0' \sim_i c_0$ distinct from both c_0 and x. Since (W, I) is irreducible and non-spherical, there exist $j \in I$ such that $m_{i,j} = \infty$. Let $c_1 \in \mathrm{Ch}(X)$ be such that the Weyl distance $\delta(c_0, c_1)$ from c_0 to c_1 is ij and that (c_0, c_0', c_1) is a minimal gallery.

By Corollary 6.3, there is $g \in N$ such that $g(c_0) = c_1$. Let σ_0 be the i-panel of c_0. For all $n > 0$, set $c_n = g^n.c_0$, $c_n' = g^n.c_0'$ and $\sigma_n = g^n.\sigma_0$.

Notice that g acts as a translation on the $\{ij\}$-residue containing x. It follows that for all $n > 0$, the gallery $(c_0, c_0', c_1, c_1', \dots, c_n, c_n')$ is minimal. Moreover for $0 \leq s < t \leq n$ we have $\mathrm{proj}_{\sigma_s}(g^t.x) = c_s'$ and $\mathrm{proj}_{\sigma_t}(g^s.x) = c_t$. Therefore, if follows from Lemma 3.3 that $X_i(g^t(x)) \subset X_i(c_t')$ and $X_i(g^s(x)) \subset X_i(c_t)$.

Since $V_i(c_t)$ and $V_i(c_t')$ commute and since $g^s V_i(x) g^{-s} = V_i(g^s.x) \leq V_i(c_t)$ and $g^t V_i(x) g^{-t} = V_i(g^t.x) \leq V_i(c_t')$, we infer that $g^s V_i(x) g^{-s}$ and $g^t V_i(x) g^{-t}$ commute. From Proposition 5.2, it follows that the various conjugates $g^n V_i(x) g^{-n}$ with $n \geq 0$ generate their direct product in $\mathrm{Aut}(X)^+$.

Recall now that g belongs to N and, hence, so does any commutator $[g, z]$ with $z \in \mathrm{Aut}(X)^+$. Given $h \in V_i(x)$, we can construct an element $z \in \prod_{n \geq 0} g^n V_i(x) g^{-n}$ satisfying the equation $[g, z] = h$ by setting $z_0 = h$ and constructing inductively the n^{th} component $z_n \in g^n V_i(x) g^{-n}$ for all $n > 0$. Thus $h \in N$ and we are done. □

7. Simplicity of the automorphism group

Proof of Theorem 1.1. Let $N \neq 1$ be a non-trivial subgroup of $G = \mathrm{Aut}(X)^+$. By Corollary 6.3, the group N is transitive on $\mathrm{Ch}(X)$. Since G is strongly transitive on X, it is naturally endowed with a BN-pair. Therefore, if we show that N contains the full stabiliser $\mathrm{Stab}_G(R)$ of some residue R, then it will follow from [Tit64, Proposition 2.2] that N itself is the stabiliser of some residue. The transitivity of N on $\mathrm{Ch}(X)$ forces that residue to be the whole building X, whence $N = G$ as desired. Therefore, the desired conclusion will follow provided we show that N contains $\mathrm{Stab}_G(R)$ of some residue R. This is the final of the following series of claims.

Claim 1. *For any residue R of irreducible type, the stabiliser* $\mathrm{Stab}_N(R)$ *maps surjectively to* $\mathrm{Aut}(R)^+$.

In order to prove the claim, we first observe that given two chambers $c, c' \in \mathrm{Ch}(R)$, any element of G mapping c to c' must stabilise R. Since N is chamber-transitive, it follows that for any residue R, the image of $N \cap \mathrm{Stab}_G(R)$ in $\mathrm{Aut}(R)$ is non-trivial.

In case R is a proper residue of irreducible non-spherical type, we infer by induction on the rank that $\mathrm{Aut}(R)^+$ is simple; notice that the base of the induction is provided by [Tit70], which settles the case of trees. Since moreover the homomorphism of $\mathrm{Stab}_G(R)$ in $\mathrm{Aut}(R)^+$ is surjective by Proposition 4.1, it follows that it remains surjective in restriction to $N \cap \mathrm{Stab}_G(R)$. In other words, we have shown that $\mathrm{Stab}_N(R)$ maps surjectively to $\mathrm{Aut}(R)^+$ for any proper irreducible non-spherical residue.

Assume now that R is spherical. Thus R is of rank one. Since (W, I) is irreducible, it follows that R is incident with a non-spherical residue R' of rank two. From the part of the claim which has already been proven, we deduce that $\mathrm{Stab}_N(R')$ maps surjectively to $\mathrm{Aut}(R')^+$. Since the latter is strongly transitive on R', it follows that $\mathrm{Stab}_N(R)$ maps surjectively on $\mathrm{Aut}(R)^+$. The claim stands proven.

Claim 2. *For any $i \in I$ and any residue R of type $i \cup i^\perp$, the group $\mathrm{Fix}_G(R)$ is contained in N.*

This was established in Lemma 6.4.

Claim 3. *Let $J = J_0 \cup J_1 \cup \cdots \cup J_s \subset I$ be the disjoint union of pairwise commuting subsets such that (W_{J_i}, J_i) is irreducible non-spherical for all $i > 0$ and (W_{J_0}, J_0) is spherical (and possibly reducible). Let $c \in \mathrm{Ch}(X)$ and $R = \mathrm{Res}_J(c)$ be its J-residue. If $\mathrm{Fix}_G(R)$ is contained in N, then so is $\mathrm{Fix}_G(\mathrm{Res}_{J_0}(c))$.*

Set $P = \mathrm{Stab}_G(R)$ and $U = \mathrm{Fix}_G(R)$. By Proposition 4.1, the quotient P/U is isomorphic to $\mathrm{Aut}(R)^+$.

For each $i = 0, \ldots, s$, set $R_i = \mathrm{Res}_{J_i}(c)$. Viewing R and each R_i as a building, we have a canonical decomposition $R \cong R_0 \times \cdots \times R_s$, which induces a corresponding product decomposition $\mathrm{Aut}(R)^+ \cong L_0 \times \cdots \times L_s$, where $L_i = \mathrm{Aut}(R_i)^+$. Let N' denote the image of N in $\mathrm{Aut}(R)^+ \cong L_0 \times \cdots \times L_s$.

Let $j > 0$. By Claim 1, the group $\mathrm{Stab}_N(R_j)$ maps surjectively to L_j. It follows that the projection of N' to L_j is surjective. Therefore, we have $[L_j, L_j] = [N', L_j] \leq N'$. Since R_j is of non-spherical type, we know that L_j is simple by induction on the rank, whence $L_j \leq N'$.

Recalling that P fits in the short exact sequence

$$1 \to U \to P \to L_0 \times \cdots \times L_s \to 1$$

and that N contains U by hypothesis, we deduce that N contains the preimage of $\{1\} \times L_1 \times \cdots \times L_s$ in P. This implies the claim, since the group

$$\mathrm{Fix}_G(R_0) = \mathrm{Ker}(\mathrm{Stab}_G(R_0) \to \mathrm{Aut}(R_0)) \leq P$$

coincides with the preimage in P of $\{1\} \times \mathrm{Stab}_{L_1}(c) \times \cdots \times \mathrm{Stab}_{L_s}(c)$.

Claim 4. *N contains the full stabiliser $\mathrm{Stab}_G(R)$ of some proper residue R.*

From Claims 2 and 3, we deduce that there exist spherical residues R_0 such that $\mathrm{Fix}_G(R_0)$ is contained in N. Amongst all such residues, we pick one, say R, whose type $J \subset I$ is of minimal possible cardinality.

If $J = \varnothing$, then R is reduced to a chamber. Thus $\mathrm{Stab}_G(R) = \mathrm{Fix}_G(R)$ is contained in N and we are done.

Assume next that J is not empty and let $j \in J$. Since (W, I) is irreducible, there exists $i \in I - J$ such that $m_{i,j} = \infty$. Now we distinguish two cases.

Assume first that $J \cup \{i\}$ is properly contained in I. Let R_i be the unique residue of type $J \cup \{i\}$ incident with R. We have $\mathrm{Stab}_G(R) \leq \mathrm{Stab}_G(R_i)$ and, hence, $N \geq \mathrm{Fix}_G(R) \geq \mathrm{Fix}_G(R_i)$. Let $R_i = R_0 \times Q_1 \times \cdots \times Q_s$ be the decomposition of R_i into a maximal spherical factor R_0 and a number of irreducible non-spherical factors. By Claim 3, we have $\mathrm{Fix}_G(R_0) \leq N$. By construction R_i is not spherical and is incident to R. Therefore the type of R_0 is a proper subset of J. This contradicts the minimality property of R, hence the present case does not occur.

Assume finally that $I = J \cup \{i\}$. Since (W, I) is irreducible, it follows that $m_{i,j'} = \infty$ for all $j' \in J$. In other words, we have $i^\perp = \varnothing$. Therefore, by Claim 2 we

have $\mathrm{Fix}_G(S) \leq N$ for any i-residue S. It follows from the minimality assumption on R that J has cardinality 1 as well. Thus $I = \{i, j\}$ and X is a tree, in which case the claim follows from the simplicity theorem in [Tit70]. □

8. Fixators of balls and residues

We now turn to fixators of balls and spherical residues. We now restrict ourselves to the locally finite case.

Proposition 8.1. *Let X be a semi-regular, locally finite right-angled building.*

Then, for all $x \in \mathrm{Ch}(X)$, $J \subseteq I$ with $W_J = \langle J \rangle$ finite and $n \geq 0$, we have the following.

(i) $\mathrm{Fix}_{\mathrm{Aut}(X)^+}(B(x, n)) = \langle U_i(c) \mid d(c, x) = n,\ \mathrm{proj}_{\mathrm{Res}_{i \cup i^\perp}(c)}(x) = c \rangle$.

(ii) $\mathrm{Fix}_{\mathrm{Aut}(X)^+}(\mathrm{Res}_J(x)) = \langle U_i(c) \mid c \in \mathrm{Ch}(\mathrm{Res}_J(x)),\ i \in I - J \rangle$.

Specialising (i) to the case $n = 0$ or (ii) to the case $J = \varnothing$, we obtain

$$\mathrm{Stab}_{\mathrm{Aut}(X)^+}(x) = \langle U_i(x) \mid i \in I \rangle.$$

Proof of Proposition 8.1. Let $G = \mathrm{Aut}(X)^+$.

(i) For all $m \geq n$, we set $G(m) = \mathrm{Fix}_G(B(x, m))$ and

$$U(m) = \langle U_i(c) \mid \mathrm{dist}(c, x) = m,\ \mathrm{proj}_{\mathrm{Res}_{i \cup i^\perp}(c)}(x) = c \rangle.$$

By Lemma 3.4, we have $B(x, m) \subset X_i(c)$ for all c such that $\mathrm{dist}(c, x) = m$ and $\mathrm{proj}_{\mathrm{Res}_{i \cup i^\perp}(c)}(x) = c$. Therefore $U_i(c)$ fixes $B(x, m)$ pointwise, whence $U(m) \leq G(m) \leq G(n)$.

We next claim that $G(m) \leq U(m)G(m+1)$. In order to see this, we pick any $h \in G(m)$. By the local finiteness of X, there are finitely only many panels $\sigma_1, \dots, \sigma_t$ with $\mathrm{Ch}(\sigma_s) \subset B(x, m+1)$ and that are not pointwise fixed by h. Since h fixes pointwise $B(x, m)$, it must fix at least one chamber in $\mathrm{Ch}(\sigma_s)$, say c_s.

Observe that $\mathrm{proj}_{\bar{\sigma}_s}(x) = c_s$. Indeed, if $\mathrm{proj}_{\bar{\sigma}_s}(x)$ were closer to x than c_s, then by Proposition 2.5(ii) the there would be a panel parallel to σ_s and entirely contained in the ball $B(x, m)$. That panel would be pointwise fixed by h, and thus h would have to act trivially on $\mathrm{Ch}(\sigma_s)$ as well, which is absurd.

Since $\mathrm{proj}_{\bar{\sigma}_s}(x) = c_s$, we deduce from Lemma 6.2 that there is $g \in U(m)$ such that gh fixes σ_s pointwise for all $s = 1, \dots, t$. Moreover Lemma 6.2 ensures that g fixes all other chambers of $B(x, m+1)$, so that $gh \in G(m+1)$. This proves the claim.

Recalling that $U(m) \leq G(m)$, we infer that for any sequence $(u_k)_{k \leq m}$ with $u_k \in U(k)$, the sequence $(u_m u_{m+1} \dots u_k)_{k \geq m}$ converges pointwise to a well defined element of $G(m)$. An inductive application of the claim provides, for each $g \in G(m)$, a sequence $(u_k)_{k \geq m}$ with $u_k \in U(k)$ such that $g = \lim_{k \to \infty} u_m u_{m+1} \dots u_k$. Therefore, the assertion (i) will follow if one shows that $U(m+1) \leq U(m)$.

Let thus $c \in \mathrm{Ch}(X)$ and $i \in I$ such that $\mathrm{dist}(c, x) = m+1$ and $\mathrm{proj}_{\mathrm{Res}_{i \cup i^\perp}(c)}(x) = c$. It suffices to show that $U_i(c)$ is contained in $U(m)$. Let c' be the first chamber on a minimal gallery from c to x, and let $j \in I$ be the type of the panel shared by c and c'. If $m_{i,j} = 2$, then the i-panel of c is parallel to the i-panel of c' since those panels are contained and opposite in a generalalized 2-gon. It then follows from Proposition 2.5(ii) that $c' \in \mathrm{Res}_{i \cup i^\perp}(c)$. Since $\mathrm{dist}(c', x) = m$, we infer that $\mathrm{proj}_{\mathrm{Res}_{i \cup i^\perp}(c)}(x) = c'$, which is absurd. Thus $m_{i,j} = \infty$, and hence we have $X_j(c') \subset X_i(c)$ by Lemma 3.3, so that $U_i(c) \leq U_j(c') \leq U(m)$ as desired.

(ii) Let $R = \mathrm{Res}_J(x)$ and set

$$U(R) = \langle U_i(c) \mid c \in \mathrm{Ch}(R),\ i \in I - J\rangle.$$

We need to show that $U(R) = \mathrm{Fix}_G(R)$. For all $c \in \mathrm{Ch}(R)$ and $i \in I - J$, we have $\mathrm{Ch}(R) \subseteq X_i(c)$ by Lemma 3.5, so that $U_i(c)$ fixes $\mathrm{Ch}(R)$ pointwise. The inclusion $U(R) \leq \mathrm{Fix}_G(R)$ follows.

We claim that $G(1) \leq U(R)$, where $G(1)$ is the pointwise stabiliser of the ball of radius 1 around c. We have seen in the proof of (i) that $G(1) = U(1)$.

Let $i \in I$ and $c \in \mathrm{Ch}(X)$ with $\mathrm{dist}(c, x) = 1$ such that $\mathrm{proj}_{\mathrm{Res}_{i \cup i^\perp}(c)}(x) = c$. Since $G(1) = U(1)$, the claim will follow if we prove that $U_i(c) \leq U(R)$. Let $j \in I$ be the type of the panel shared by c and x. The condition that $\mathrm{proj}_{\mathrm{Res}_{i \cup i^\perp}(c)}(x) = c$ implies $m_{i,j} = \infty$.

If $j \in J$, then $c \in \mathrm{Ch}(R)$ and $i \notin J$ (since W_J is finite), so that $U_i(c) \leq U(R)$ by the definition of $U(R)$.

If $j \notin J$, we have $U_i(c) \leq U_j(x)$ by Lemma 3.3 and $U_j(x) \leq U(R)$ by the definition of $U(R)$. We conclude again that $U_i(c) \leq U(R)$, and the claim stands proven.

Let now $h \in \mathrm{Fix}_G(R)$. Let $i_1, \dots, i_t$ be all the elements of $I - J$. By Lemma 6.2, there is $g \in \langle U_{i_s}(x) \mid s = 1, \dots, t\rangle \leq U(R)$ such that gh fixes pointwise the ball of radius 1 around x. Thus $gh \in G(1) \leq U(R)$ and, hence $h \in U(R)$, as desired. □

9. Local structure

A **locally normal** subgroup of a locally compact group is a compact subgroup whose normaliser is open. We first record that automorphisms of right-angled buildings always admit many locally normal subgroups.

Lemma 9.1. *Let X be a thick, semi-regular, locally finite, right-angled building of type (W, I). Assume that (W, I) is irreducible non-spherical.*

Then $\mathrm{Aut}(X)^+$ admits locally normal subgroups which decompose non-trivially as direct products, all of whose factors are themselves locally normal.

Proof. Given $c \in \mathrm{Ch}(C)$ and $i \in I$, the group $V_i(c)$ is closed by definition, compact because it fixes c, and non-trivial by Lemma 5.1. It is moreover normalised by the pointwise stabiliser of the i-panel of c, which is open since X is locally finite. This proves that $V_i(c)$ is a locally normal subgroup. The desired conclusion is thus provided by Proposition 5.2. □

The following result is an extended version of Theorem 1.3 from the introduction.

Theorem 9.2. *Let X be a thick, semi-regular, locally finite, right-angled building of type (W, I). Assume that (W, I) is irreducible non-spherical.*

Then the following are equivalent.

(i) W is one-ended.
(ii) W does not split as a free amalgamated product over a finite subgroup.
(iii) There is no partition $I = I_0 \cup I_1 \cup I_2$ with I_1, I_2 non-empty, $m_{i,j} = 2$ for all $i, j \in I_0$ and $m_{i,j} = \infty$ for all $i \in I_1$ and $j \in I_2$.
(iv) X is one-ended.
(v) G is one-ended.
(vi) All compact open subgroups of $G = \mathrm{Aut}(X)^+$ are indecomposable.

We shall need the following basic fact on right-angled Coxeter groups.

Lemma 9.3. *Let (W, I) be an irreducible non-spherical right-angled Coxeter system.*

For any two half-spaces H, H' whose boundary walls cross in the Davis complex of W, there is a half-space H'' properly contained in $H \cap H'$.

Proof. The Davis complex of a right-angled Coxeter group is a CAT(0) complex. By [CS11, Lem. 5.2], at least one of the four sectors determined by the boundary walls of H and H' properly contains a half-space. Transforming that half-space by an appropriate element from the group generated by the reflections fixing H and H', we find a half-space properly contained in $H \cap H'$, as desired. □

Proof of Theorem 9.2. The equivalences (i) ⇔ (ii) ⇔ (iii) are well-known, see [MT09]. The equivalence (iv) ⇔ (v) is clear since G acts properly and cocompactly on X, so that G and X are quasi-isometric.

(i) ⇒ (iv) By assumption all apartments are one-ended. Given $x \in \mathrm{Ch}(X)$, we need to prove that for all $n \geq 0$, any two chambers c', c'' at distance $> n$ away from x can be connected by a gallery avoiding the ball $B(x, n)$. We proceed by induction on n.

In the base case $n = 0$, either a minimal gallery from c' to c'' does not pass through x, and we are done, or every apartment containing c' and c'' also contains x, in which case we can find a gallery from c' to c'' avoiding x inside one of these apartments, since these are one-ended by hypothesis.

Let now $n > 0$ and assume that $\mathrm{Ch}(X) - B(x, n-1)$ is gallery-connected. Let $c' = c_0, c_1, \dots, c_t = c''$ be a gallery from c' to c'' which does not meet $B(x, n-1)$. Then for all i, if $c_i \in B(x, n)$ then $\mathrm{dist}(c_{i-1}, x) = \mathrm{dist}(c_{i+1}, x) = n + 1$. Therefore, it suffices to prove that if $\mathrm{dist}(c', x) = \mathrm{dist}(c'', x) = n$ and c', c'' are both adjacent to a common chamber $d \in B(x, n-1)$, then there is a gallery from c' to c'' avoiding $B(x, n)$. Let Σ be an apartment containing x and d. Let d' and d'' be the two chambers of Σ different from d and respectively sharing with d the common panel of d and c', and of d and c''. Let i' (resp. i'') be the type of the panel σ' (resp. σ'') shared by d, d' and c' (resp. d, d'' and c''). Clearly $\mathrm{proj}_{\bar{\sigma}'}(d) = d$ and $\mathrm{proj}_{\bar{\sigma}''}(d) = d$. Therefore, by Lemma 6.2, there is an element $g \in G$ fixing $X_{i'}(d) \cap X_{i''}(d)$ pointwise and such that $g(d') = c'$ and $g(d'') = c''$. Since $x \in X_{i'}(d) \cap X_{i''}(d)$, it follows that $g(\Sigma)$ is an apartment containing x, d, c' and c''. Since apartments are one-ended, a gallery joining c' to c'' and avoiding $B(x, n)$ can be found in the apartment $g(\Sigma)$, and we are done.

(v) ⇒ (iii) Assume that (iii) fails and let $I = I_0 \cup I_1 \cup I_2$ be a partition with I_1, I_2 non-empty, $m_{i,j} = 2$ for all $i, j \in I_0$ and $m_{i,j} = \infty$ for all $i \in I_1$ and $j \in I_2$. Let T be the graph whose vertex set is the collections of residues of type $I_0 \cup I_1$ and $I_0 \cup I_2$, and declare that two residues are adjacent if they contain a common residue of type I_0. By Lemma 4.3 from [HP03] the graph T is a tree. Since $\langle I_0 \rangle$ is finite and since X is locally finite, the residues of type I_0 are finite and, hence, their stabilisers are compact open subgroups. In other words the edge stabilisers of the tree T are compact open subgroups. Since G is chamber-transitive, it acts edge-transitively on T and, hence, G cannot be one-ended by [Abe74].

(vi) ⇒ (iii) Assume that (iii) fails and let $I = I_0 \cup I_1 \cup I_2$ be a partition with I_1, I_2 non-empty, $m_{i,j} = 2$ for all $i, j \in I_0$ and $m_{i,j} = \infty$ for all $i \in I_1$ and $j \in I_2$. Let R be a residue of type I_0 in X. Since $\langle I_0 \rangle$ is finite, the set $\mathrm{Ch}(R)$ is finite and, hence, the pointwise stabiliser $\mathrm{Fix}_G(R)$ is a compact open subgroup. We shall prove that $\mathrm{Fix}_G(R)$ splits non-trivially as a direct product.

For $k = 1, 2$, let $U_k = \langle U_i(c) \mid c \in \mathrm{Ch}(R), i \in I_k \rangle$. Notice that U_1 and U_2 are both non-trivial since I_1 and I_2 are assumed non-empty.

By Proposition 8.1, we have $\mathrm{Fix}_G(R) = \langle U_1 \cup U_2 \rangle$. We claim that U_1 and U_2 commute. Indeed, let $c_1, c_2 \in \mathrm{Ch}(R)$, let $i_1 \in I_1$, $i_2 \in I_2$. It suffices to prove that $U_{i_1}(c_1)$ and $U_{i_2}(c_2)$ commute. This in turn will follow if one shows that they have disjoint support.

By definition the support of $U_{i_1}(c_1)$ is the union of the sets $X_{i_1}(d)$ over all chambers d that i_1-adjacent to but different from c_1. Let d be such a chamber. We claim that $X_{i_1}(d) \subset X_{i_2}(c_2)$.

By Lemma 3.5, we have $c_2 \in X_{i_1}(c_1)$ so that $c_2 \notin X_{i_1}(d)$. Similarly, Lemma 3.5 implies that $c_1 \in X_{i_2}(c_2)$, which implies that $d \in X_{i_2}(c_2)$, since otherwise a panel of type i_1 would be parallel to a panel of type i_2 by Corollary 2.4, which is impossible by Proposition 2.5(i). This proves that $d \in X_{i_2}(c_2)$ and $c_2 \notin X_{i_1}(d)$. The claim then follows from Lemma 3.3.

The claim implies that the support of $U_{i_1}(c_1)$ is pointwise fixed by $U_{i_2}(c_2)$. By symmetry, the support of $U_{i_2}(c_2)$ is pointwise fixed by $U_{i_1}(c_1)$, so that $U_{i_1}(c_1)$ and $U_{i_2}(c_2)$ commute, as desired. This confirms that U_1 and U_2 commute.

Since $U_1.U_2 = \mathrm{Fix}_G(R)$, we have $U_1 \cap U_2 \leq \mathscr{Z}(\mathrm{Fix}_G(R))$. Hence $U_1 \cap U_2$ is contained in the **quasi-centre** of G, i.e. the collection of elements commuting with an open subgroup. By [BEW11, Theorem 4.8] the group G has trivial quasi-centre since G is compactly generated and simple. Thus $\mathrm{Fix}_G(R) \cong U_1 \times U_2$ as desired.

(iv) $\Rightarrow$ (vi). Assume finally that (iv) holds and let $U \leq G$ be a compact open subgroup with two commuting subgroups A, B such that $U = A.B$. We shall prove that $\overline{A}$ or $\overline{B}$ is open. Since the closures $\overline{A}$ and $\overline{B}$ commute, we infer that B or A is in the quasi-centre of G, which is trivial by [BEW11, Theorem 4.8] since G is simple. Thus $U = A$ or $U = B$ and (v) holds.

Therefore, all we need to show that if $U = A.B$ is the commuting product of two closed subgroups, then A or B is open; equivalently we need to show that A or B is finite. This follows from the last of a series of claims which we shall now prove successively.

Let $x \in \mathrm{Ch}(X)$. Upon replacing A and B by their respective intersection with the compact open subgroup $\mathrm{Stab}_G(x)$ and then redefining U accordingly, we may assume that U fixes x. For all $m \geq 0$, we set $G(m) = \mathrm{Fix}_G(B(x, m))$. Since U is open it contains $G(n_0)$ for some $n_0 \geq 0$. We define

$$\Pi = \{\sigma \text{ panel of } X \mid \mathrm{Stab}_{G(n_0)}(\sigma) \not\leq \mathrm{Fix}_G(\sigma)\}.$$

In particular, if $\sigma \in \Pi$ then $\mathrm{dist}(\sigma, x) \geq n_0$.

Moreover, to each chamber $c \in \mathrm{Ch}(X)$, we associate two subsets of I defined as follows

$$I_0(c) = \{i \in I \mid \mathrm{proj}_{\mathrm{Res}_i(c)}(x) \neq c\}$$

and

$$I_\Pi(c) = \{i \in I \mid \mathrm{Res}_i(c) \in \Pi\}.$$

Recall that a subset $J \subseteq I$ is called **spherical** if it generates a finite subgroup of W. It is a classical fact that $I_0(c)$ is a spherical subset of I.

Claim 1. *Let $c \in \mathrm{Ch}(X)$ with $\mathrm{dist}(c, x) > n_0$ and $i \in I$. Then $i \in I_\Pi(c)$ if and only if $\mathrm{dist}(x, \mathrm{Res}_{i \cup i^\perp}(c)) \geq n_0$.*

Let σ be the i-panel of c, let $\bar{\sigma}$ be the $i \cup i^\perp$-residue of c and $c' = \mathrm{proj}_{\bar{\sigma}}(x)$.

If $\mathrm{dist}(x, \bar{\sigma}) = \mathrm{dist}(x, c') \geq n_0$, then $U_i(c')$ fixes $B(x, n_0)$ pointwise by Lemma 3.4. Thus $U_i(c') \leq G(n_0) \leq U$. Since σ is parallel to the i-panel of c' by Proposition 2.5(ii), we infer that $U_i(c')$ fixes $\mathrm{proj}_\sigma(x)$ and permutes arbitrarily all the other chambers of σ. Therefore $\mathrm{Stab}_{G(n_0)}(\sigma) \not\leq \mathrm{Fix}_G(\sigma)$. Thus $\sigma \in \Pi$ and $i \in I_\Pi(c)$.

Assume conversely that $\mathrm{dist}(x, \bar{\sigma}) < n_0$. Then the i-panel of c' lies entirely in $B(x, n_0)$ and it thus pointwise fixed by $G(n_0)$. Therefore $\mathrm{Stab}_{G(n_0)}(\sigma)$ acts trivially on σ, and hence $\sigma \notin \Pi$ and $i \notin I_\Pi(c)$.

Claim 2. *There is some $n_1 > n_0$ such that for all $c \in \mathrm{Ch}(X)$ with $\mathrm{dist}(c, x) > n_1$, we have $I_0(c) \cap I_\Pi(c) \neq \varnothing$.*

Since (W, I) is right-angled, any collection of pairwise intersecting walls in an apartment is contained in the set of walls of a spherical residue. The cardinality of such a collection is bounded above by the largest cardinality of a spherical subset of I. In particular it is finite. In view of Ramsey's theorem, we infer that there is some $n_1 > 0$ such that any set of more than n_1 walls contains a subset of more than n_0 pairwise non-intersecting wall.

Let now $c \in \mathrm{Ch}(X)$ with $\mathrm{dist}(c, x) > n_1$ and Σ be an apartment containing c and x. By construction there is a set of more than n_0 pairwise non-intersecting walls in Σ that are crossed by any minimal gallery from c to x. In particular, at least one of these walls, say $\mathcal{W}$, separates c from the ball $B(x, n_0)$.

Since (W, I) is right-angled, no wall crossed by a shortest possible gallery from c to a chamber adjacent to $\mathcal{W}$ crosses $\mathcal{W}$. Let $\mathcal{W}'$ be the first wall crossed by such a gallery. Thus $\mathcal{W}'$ adjacent to c and separates c from the ball $B(x, n_0)$.

Let now $k \in I$ be the type of the panel of c which belongs to $\mathcal{W}'$. Since $\mathcal{W}'$ separates c from x, we have $k \in I_0(c)$. Since $\mathcal{W}'$ does not meet the ball $B(x, n_0)$, we have $\mathrm{dist}(x, \mathrm{Res}_{k \cup k^\perp}(c)) > n_0$ because $\mathrm{dist}(x, \mathrm{Res}_{k \cup k^\perp}(c))$ belongs to Σ and is bounded by the wall $\mathcal{W}'$. Therefore $k \in I_\Pi(c)$ by Claim 1. Thus the sets $I_0(c)$ and $I_\Pi(c)$ have a non-empty intersection, as desired.

Claim 3. *Let $c \in \mathrm{Ch}(X)$ and σ be a panel of c. If $a(c) \neq c$ for some $a \in \mathrm{Stab}_A(\sigma)$, then $b(c) = c$ for all $b \in \mathrm{Stab}_B(\sigma)$, and similarly with A and B interchanged.*

Let $i \in I$ be the type of σ. Notice that $\mathrm{proj}_\sigma(x) \neq c$ since a fixes x and stabilises σ. Since (W, I) is irreducible and non-spherical, there is $j \in I$ such that $m_{i,j} = \infty$. Given a chamber c' in the $\{i, j\}$-residue of c such that $\mathrm{proj}_\sigma(c') = c$, we deduce from Lemma 3.2 that c' is separated from x by at least $\mathrm{dist}(c, c')$ wall-residues of type $i \cup i^\perp$ or $j \cup j^\perp$. In particular, we may find such a c' with the property that $X_i(c')$ does not intersect the ball $B(x, n_0)$. Moreover, up to replacing c' by an i-adjacent chamber, we can choose c' so that $c \notin X_i(c')$. Consequently the group $V_i(c')$ fixes $B(x, n_0)$ pointwise, and is thus contained in U.

Lemma 3.3 ensures that $X_i(c') \subset X_i(c)$, whence $V_i(c') \leq V_i(c)$. By Proposition 5.2, the conjugate $aV_i(c')a^{-1} \leq V_i(a(c))$ commutes with $V_i(c')$. Therefore the commutator $[a, V_i(c')]$ is a subgroup isomorphic to $V_i(c')$ which is diagonally embedded in $V_i(c') \times V_i(a(c'))$, and its action on the wing $X_i(c')$ is given by a surjective map onto $V_i(c')$. Moreover, since A is normal in U, we also have $[a, V_i(c')] \leq A$.

Similarly, if $b \in \mathrm{Stab}_B(\sigma)$ and $b(c) \neq c$, then we have $[b, V_i(b(c'))] \leq B$ and $[b, V_i(c')]$ maps surjectively on $V_i(c')$. Since A and B commute, it follows that $V_i(c')$ is abelian, in contradiction with Lemma 5.1. Therefore $b(c) = c$ for all $b \in \mathrm{Stab}_B(\sigma)$.

Claim 4. *For each panel $\sigma \in \Pi$, there is a unique $F \in \{A, B\}$ with $\mathrm{Stab}_F(\sigma) \not\leq \mathrm{Fix}_G(\sigma)$. We denote the corresponding function by*

$$f \colon \Pi \to \{A, B\} \colon \sigma \mapsto F.$$

Moreover, the group $\mathrm{Stab}_F(\sigma)$ permutes arbitrarily the elements of $\mathrm{Ch}(\sigma)$ different from $\mathrm{proj}_\sigma(x)$ (i.e. it induces the full symmetric group on $\mathrm{Ch}(\sigma) - \{\mathrm{proj}_\sigma(x)\}$).

Let $\sigma \in \Pi$. By definition there is $u \in \mathrm{Stab}_{G(n_0)}(\sigma)$ and $c \in \mathrm{Ch}(\sigma)$ with $u(c) \neq c$. Write $u = ab$ with $a \in A$ and $b \in B$. Consider a gallery $x_0, x_1, \dots, x_k$ of minimal possible length joining a chamber in $B(x, n_0)$ to a chamber in $\mathrm{Ch}(\sigma)$. Since $u \in G(n_0)$, it fixes x_0 and by the minimality of the gallery, we have $x_k = \mathrm{proj}_\sigma(x_0)$, so that u fixes x_k as well. Therefore u fixes x_i for all i.

We claim that a and b both fix x_i for all i. Otherwise there is some i such that $a(x_i) \neq x_i$. Since $u(x_i) = x_i$ and $u = ab$, we must have $b(x_i) \neq x_i$. If i the smallest such index, then a and b also fix x_{i-1} and thus both stabilise the panel shared by x_{i-1} and x_i. This contradicts the previous claim.

It follows that a and b both fix x_k and hence stabilise σ. In particular we have $\mathrm{Stab}_A(\sigma) \not\leq \mathrm{Fix}_G(\sigma)$ or $\mathrm{Stab}_B(\sigma) \not\leq \mathrm{Fix}_G(\sigma)$. It remains to show that these two possibilities are mutually exclusive. Let Ch_A and Ch_B be the subsets of $\mathrm{Ch}(\sigma)$ that are not fixed by A and B respectively. The previous claim guarantees that Ch_A and Ch_B are disjoint.

Let $i \in I$ be the type of σ and $c' = \mathrm{proj}_\sigma(x)$. Since $\sigma \in \Pi$, we have $U_i(c') \leq G(n_0) \leq U$ by Claim 1 and Lemma 3.4. Consequently the group $\mathrm{Stab}_U(\sigma)$ permutes arbitrarily the set $\mathrm{Ch}(\sigma) - \{c'\}$ by Proposition 4.2. Since A and B are normal in U, it follows that either Ch_A or Ch_B coincides with the whole of $\mathrm{Ch}(\sigma) - \{c'\}$.

Claim 5. *Let $c \in \mathrm{Ch}(X)$ and $i, j \in I$ with $m_{i,j} = 2$. Let σ_i and σ_j be the i- and j-panels of c respectively. If σ_i and σ_j belong to Π, then $f(\sigma_i) = f(\sigma_j)$.*

Suppose for a contradiction that $f(\sigma_i) = A$ and $f(\sigma_j) = B$. Then there are some $a \in A$, $b \in B$ stabilising respectively σ_i and σ_j, and so that moreover $a(c_i) \neq c_i$ and $b(c_j) \neq c_j$ for some $c_i \in \mathrm{Ch}(\sigma_i)$ and $c_j \in \mathrm{Ch}(\sigma_j)$.

Let R be the $\{i, j\}$-residue of c and set $c' = \mathrm{proj}_R(x)$. Let also σ'_i and σ'_j be the i- and j-panels of c'. Then a and b both fix c' and stabilise σ'_i and σ'_j. Moreover σ'_i and σ'_j are respectively parallel to σ_i and σ_j. Therefore we have $f(\sigma_i) = f(\sigma'_i)$ and $f(\sigma_j) = f(\sigma'_j)$.

Let Σ be an apartment containing x and c'. By Claim 1, the ball $B(x, n_0)$ is contained in $X_i(c') \cap X_j(c')$. From Lemma 6.2, we deduce that there is some $g \in G(n_0) \leq U$ mapping Σ to an apartment containing c_i and c_j. Upon replacing Σ by $g(\Sigma)$, we can thus assume that Σ is an apartment containing the chambers x, c, c_i and c_j.

Let H (resp. H') be the half-apartment of Σ containing c_i (resp. c_j) but not c. Since (W, I) is irreducible and non-spherical, there is a half-apartment H'' which is entirely contained in $H \cap H'$ by Lemma 9.3. Let c'' be a chamber of H'' having a panel in the wall determined by H'', and let $k \in I$ be the type of that panel. Since $H'' \subset H \cap H'$, we deduce from Lemma 3.3 that $X_k(c'') \subseteq X_i(c_i) \cap X_j(c_j)$. In particular we have $V_k(c'') \leq V_i(c_i) \cap V_j(c_j) \leq G(n_0) \leq U$ (see Lemma 3.4).

By Proposition 5.2, the commutator $[a, V_i(c_i)]$ is a subgroup isomorphic to $V_i(c_i)$ which is diagonally embedded in $V_i(c_i) \times V_i(a(c_i))$, and its action on the wing $X_i(c_i)$ is provided by a surjection onto $V_i(c_i)$. Moreover, since A is normal in U, we also

have $[a, V_i(c_i)] \leq A$. Similarly we have $[b, V_j(c_j)] \leq B$ and its action on the wing $X_j(c_j)$ is provided by a surjection on $V_j(c_j)$.

This implies that the groups $[a, V_k(c'')] \leq A$ and $[b, V_k(c'')] \leq B$ are two commuting subgroups, both of which map surjectively onto $V_k(c'')$. It follows that $V_k(c'')$ is abelian, in contradiction with Lemma 5.1. The claim stands proven.

Claim 6. *Let $c \in \mathrm{Ch}(X)$ and $i, j \in I$ with $m_{i,j} = \infty$. Let σ_i and σ_j be the i- and j-panels of c respectively. If σ_i and σ_j belong to Π, and if $\mathrm{proj}_{\sigma_i}(x) \neq c$, then $f(\sigma_i) = f(\sigma_j)$.*

Suppose for a contradiction that $f(\sigma_i) = A$ and $f(\sigma_j) = B$ (the case $f(\sigma_j) = A$ and $f(\sigma_i) = B$ is treated similarly). In view of Claim 4 and the fact that $c' = \mathrm{proj}_{\sigma_i}(x) \neq c$, we can find $a \in A$, $b \in B$ and $c_j \in \mathrm{Ch}(\sigma_j)$ such that $a(c) \neq c$ and $b(c_j) \neq c_j$.

By Claim 1 the ball $B(x, n_0)$ is contained in $X_i(c')$. By Lemma 3.3 we have $X_i(c) \supset X_j(c_j)$. In particular $X_j(c_j)$ is disjoint from $B(x, n_0)$, from which it follows that $V_j(c_j)$ is contained in U. Arguing as before, we conclude that the groups $[a, V_j(c_j)]$ and $[b, V_j(c_j)]$ commute and map onto $V_j(c_j)$, forcing the latter to be abelian, in contradiction with Lemma 5.1.

Claim 7. *Let $c \in \mathrm{Ch}(X)$, let $i, j \in I$ and let σ_j be the i- and j-panels of c respectively. If σ_i and σ_j belong to Π, and if $\mathrm{dist}(c, x) > n_1$, then $f(\sigma_i) = f(\sigma_j)$.*

It suffices to deal with the case when $\mathrm{proj}_{\sigma_i}(x) = \mathrm{proj}_{\sigma_j}(x) = c$, since the other cases are dealt with by Claims 5 and 6.

Since $\mathrm{dist}(c, x) > n_1$, there is some $k \in I_0(c) \cap I_\Pi(c)$ by Claim 2. Let σ_k be the k-panel of c. Invoking Claim 5 or Claim 6 according as $m_{ik} = 2$ or $m_{ik} = \infty$, we infer that $f(\sigma_i) = f(\sigma_k)$. Similarly $f(\sigma_j) = f(\sigma_k)$, so that $f(\sigma_i) = f(\sigma_j)$ and we are done.

Notice that by Claim 2, every chamber c at distance $> n_1$ from x has a panel belonging to Π. Moreover the map f takes the same value on all these panels by Claim 7. We shall denote this common value by $f(c)$.

Claim 8. *Let $c, c' \in \mathrm{Ch}(X)$ be two adjacent chambers both at distance $> n_1$ from x. Then $f(c) = f(c')$.*

Let σ be the panel shared by c and c'. If $\sigma \in \Pi$ then we are done by the previous claim. We assume henceforth that $\sigma \notin \Pi$ and denote by j its type. By Claim 2 there is some $i \in I_0(c) \cap I_\Pi(c)$. Let σ_i be the i-panel of c. Then $d = \mathrm{proj}_{\sigma_i}(x)$ is different from c and moreover $\sigma_i \in \Pi$. By Claim 1 this implies that $B(x, n_0)$ is entirely contained in $X_i(d)$. It follows that $m_{i,j} = 2$, since otherwise we would have $X_i(d) \subset X_j(c)$ by Lemma 3.3 and hence the $j \cup j^\perp$-wall residue of c would not meet $B(x, n_0)$. This would contradict Claim 1 since $\sigma \notin \Pi$.

Since $m_{i,j} = 2$, it follows that the i-panel of c', say σ'_i, is parallel to σ_i since they are contained and opposite in the $\{i, j\}$-residue of c. Therefore, any element of $G(n_0) \leq U$ stabilises σ_i and acts non-trivially on it if and only if it stabilises σ'_i and acts non-trivially on it. Hence we find that $f(\sigma_i) = f(\sigma'_i)$ whence $f(c) = f(c')$.

Claim 9. *We have $A \cap G(n_1) = 1$ or $B \cap G(n_1) = 1$.*

By (iv) any two chambers at distance $> n_1$ from x can be joined by a gallery which does not meet the ball $B(x, n_1)$. By the preceding claim, this implies that the map f is constant on $\mathrm{Ch}(X) - B(x, n_1)$. Upon exchanging A and B we may assume

that this constant value is A. It follows that for all panels $\sigma \in \Pi$ at distance $> n_1$ from x, we have $\mathrm{Stab}_B(\sigma) \leq \mathrm{Fix}_B(\sigma)$. An immediate induction now shows that for all $m > n_1$, we have $B \cap G(m) \leq G(m+1)$, whence $B \cap G(n_1)$ is trivial. □

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